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DECIDABLE MODELS OF SMALL THEORIES

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ABSTRACT. Many counterexamples are known in the class of small
theories due to Goncharov [3] and Millar [5]. The prime model of a de-
cidable small theory is not necessarily decidable. The saturated model
of a hereditarily decidable small theory is not necessarily decidable. A
homogeneous model with uniformly decidable type spectra is not neces-
sarily decidable. In this paper, I consider the questions of what model
theoretic properties are sufficient for the existence of such counterexam-
ples. I introduce a subclass of the class of small theories, which I call AL
theories, show the absence of Goncharov-Millar counterexamples in this
class, and isolate a model theoretic property that implies the existence
of such anomalies among computable models.

I consider only countable structures of countable languages. And only
small theories.

Definition 1. A first-order theory T is *small* if the set of finite first-order
types of T without parameters, $S(T)$, is countable.

Throughout the paper all theories are assumed to be small.

Fact 1. *Every small theory has a prime model and a saturated model.
These models are unique up to isomorphism.*

Fact 2. *If p is a type of a small theory T and $A \models p(\bar{a})$ then the theory
$Th(A, \bar{a})$ has a prime model $(A_{\bar{a}}, \bar{c})$. The isomorphism type of the struc-
ture $A_{\bar{a}}$ does not depend on the choice of A and $\bar{a}$, it only depends on the*

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type p. Since **we consider structures up to isomorphism***, denote the structure by A_p.*

Definition 2. Call the structure A_p from Fact 2 *p-prime*, or *almost prime* if the type is not specified.

The set $\mathcal{AP}_T$ of all almost prime models of a theory T is preordered by the relation $\preceq$ of elementary embeddability. The preorder induces a partial order on the factor-set $\mathcal{AP}_T/\sim$, where $A \sim B \Leftrightarrow (A \preceq B \,\&\, B \preceq A)$, in the natural way. Note that $(\mathcal{AP}_T/\sim, \preceq)$ has a unique least element—the prime model of T.

Definition 3. We call the partial order $(\mathcal{AP}_T/\sim, \preceq)$ the *fundamental order* of the theory T.

The notion of fundamental order is also known as Rudin–Keisler order [6, 7]. In this paper, I follow the terminology of Lascar–Poizat [4] and Baldwin–Berman [1], who introduced and studied fundamental orders of theories in general.

To illustrate the definition, let us consider the following simple though important examples of fundamental orders.

Example 1. *The following theories T are AL theories in the sense of Definition 5 below.*

(1) *A saturated structure is almost prime if and only if the structure is $\aleph_0$-categorical. For the theory T of such a structure, $\mathcal{AP}_T$ is the one-element partial ordering.*

(2) *If a theory T is $\aleph_1$- but not $\aleph_0$-categorical then $\mathcal{AP}_T \cong \omega$.*

(3) *If a complete theory T has finitely many countable models then $\mathcal{AP}_T$ has a greatest element.*

Proof. (1) is obvious. For (2) we note that the saturated model of such a theory is not almost prime and the rest of models are almost prime.

(3) A type p of a theory T is called powerful if every model of T realising p realises every type of T as well. If T is a complete theory with finitely many, but more than one, countable models then it has a non-principal powerful type p. A p-prime structure gives the greatest element of $\mathcal{AP}_T$. $\square$

Proposition 1 ([6]). *If $A_p \sim A_q$ but $A_p \not\cong A_q$, then there is a structure A such that $A \sim A_p$ but A is not almost prime.*

Proof. Form an elementary chain $A_0 \subseteq A_1 \subseteq \ldots$ where $A_n \cong A_p$ if n is even and $A_n \cong A_q$ if n is odd. Put $A = \bigcup_{n\in\omega} A_n$. $\square$

The structure A can be presented as a union of an elementary chain of isomorphic almost prime structures, but A itself is not almost prime. Following Sudoplatov [6, 7], I shall call such a structure limit:

Definition 4. A structure is *p-limit* if it is a union of an elementary chain of p-prime structures but it itself is not p-prime. A structure is *limit* if it is a union of an elementary chain of isomorphic almost prime structures but it itself is not almost prime.

Definition 5. A complete small theory T is an *AL theory* if every countable non-saturated model of T is either almost prime or limit.

Definition 5 can be rewritten syntactically [6].
Example 1 above gives some well-known examples of AL theories.
Note that a saturated structure is limit if and only if its theory has a non-principal powerful type, i.e. $\mathcal{AP}_T$ has a maximal element.
Let $\mathcal{LS}_T$ denote a set of all limit models of an AL theory T. The structure of the spectrum of models of the theory T is determined by a pre-ordering $\mathcal{AP}_T$ and a function $\lambda_T : \mathcal{AP}_T \to 2^{\mathcal{LS}_T}$ mapping a p-prime structure to the set of all p-limit structures [6, 7]. Think of λ_T as of a disjoint union of bipartite graphs. $\mathcal{LS}_T = \bigcup\limits_{M \in \mathcal{AP}_T} \lambda_T(M)$.

Question 1. *How distinct are the class of AL theories and the class of small theories?*

Naturally, the class of AL theories is a proper subclass of the class of small theories. The following definition isolates a model theoretic property of small theories that might be violated in the class of AL theories. Sudoplatov proved in [6] that if we relax Definition 5 using Definition 6 then we get the class of all small theories.

Definition 6 ([7]). A structure is *weakly limit* if it is the union of an elementary chain of almost prime structures.

For the sake of completeness, I include the proof of Sudoplatov's lemma [6] that every countable model of a small theory is either almost prime or weakly limit.

Lemma 1 (S. Sudoplatov [6]). *Every countable model of a small theory is either almost prime or weakly limit.* □

Proof. Let A be a countable model of a small theory T. Let $a_0, a_1, \ldots$ be an enumeration of the domain of A. Consider the type $tp(a_0)$ of a_0 in A. Since T is small, there is a prime model (A_0, a_0) of $tp(a_0)$ in

the signature enriched by one constant. We can choose A_0 to be an elementary submodel of A. If A_0 is isomorphic to A, we are done and A is almost prime. If not, take the first element b_1 in our enumeration $a_0, a_1, \ldots$ such that $b_1 \in A \setminus A_0$. Consider a prime model (A_1, a_0, b_1) of $tp(a_0, b_1)$. Again, if A_1 is isomorphic to A then A is almost prime. If not, we continue the process. If the process terminates after finitely many steps, the model A is prime over $(a_0, b_1, \ldots, b_n)$ for some n and hence is almost prime. If not, we obtain an elementary chain $A_0 \preceq A_1 \preceq \ldots$ of almost prime structures. Since $a_0, a_1, \ldots$ is an enumeration of all elements of A, we have that $\cup_i A_i = A$ and A is weakly limit. $\square$

The following general theorem shows that the class of AL theories is rather different from the class of all small theories if we study decidable models. This theorem is the key property to the absence of Goncharov-Millar counterexamples in the class of AL theories.

Theorem 1. *Let T be an AL theory. Then the set of decidable almost prime models of T forms an ideal in the fundamental order of T.*

Proof. Let us first note the following property of decidable models of an AL theory, which is interesting on its own.

Lemma 2. *Let T be an AL theory and A its non-saturated decidable model. Then there exist a type p and a decidable p-prime model A_p such that the type spectra of the structures A and A_p coincide.*

Proof. If A is almost prime, there is nothing to prove. Suppose A is p-limit for some p. Since A is decidable, the set of types realised in A is uniformly computable. This gives a uniformly computable enumeration of principal types realised in $(A_{\bar{a}}, \bar{c})$, where $\bar{a}$ is a realisation of p in A. Hence, $(A_{\bar{a}}, \bar{c})$ is decidable and so is $A_{\bar{a}}$. It remains to note that the type spectra of A and $A_{\bar{a}}$ coincide. $\square$

I now prove that the set of decidable models of an AL theory is directed upwards.

Lemma 3. *Let T be an AL theory and A and B be its decidable models. Then there exists a decidable model C such that $C \succeq A$ and $C \succeq B$.*

Proof. Using Lemma 2 above, one can assume that A is a p-prime and B is a q-prime structure for some decidable types p and q. Take C to be a decidable structure that realises both p and q. Such a structure exists because otherwise the theory T would have more than countably many types. $\square$

To finish the proof of the theorem, we need to show that the set of decidable almost prime models of T is closed downwards. Let A_p be a decidable almost prime model of T and A_q be an almost prime model of T such that $A_q \preceq A_p$. We want to show that A_q is decidable. Since T is AL, there are only finitely many almost prime structures with pairwise distinct type spectra $A_0 = A_q, A_1, \ldots, A_n = A_p$ such that $A_0 \preceq A_1 \preceq \ldots \preceq A_n$. Using Millar's type omitting theorem [5] (see also [3]), we can realise all the types that are realised in A_q and omit all the types that are not realised in A_q, in a decidable structure B. That is, the structure B has the same type spectra as A_q and is decidable. We apply Lemma 2 to prove that A_q is itself decidable. Since the set of decidable almost prime models of the theory T is directed upwards and closed downwards, it forms an ideal. $\square$

Corollary 1. *If an AL theory T is decidable then T has a decidable prime model.*

Proof. Since the theory T is decidable, it has a decidable model A. If the model A is saturated then the prime model of T is decidable, see [3]. If the model A is not saturated then it is p-prime or p-limit and we apply Theorem 1 to prove that the prime model of T is decidable. $\square$

Note that Corollary 1 does not hold in the class of all small theories. In [2], we construct a decidable small theory T whose all types are decidable yet whose prime model is not decidable. Millar [5] was the first to construct such an example but his construction uses an infinite language in an essential way, while our structure is a graph. Since the prime model of a theory with decidable saturated model is decidable, we have an example of a decidable small theory whose saturated model is not decidable.

Although the class of AL theories and the class of small theories have similar model theoretic properties, they are significantly different from the computability point of view. Corollary 1 along with the following corollary demonstrate this difference.

Corollary 2. *If T is a decidable AL theory whose types are all decidable, then every almost prime model of T is decidable.*

Proof. Let A_p be an almost prime model. Since p is a decidable type, it can be realised is a decidable model. Applying Theorem 1 we get that A_p is decidable. $\square$

In the light of Theorem 1, a natural question to ask would be what ideals can be formed by decidable models? The following theorem of

ours answers this question in full for the class of theories with lattice-like fundamental orders. The question remains open in the general case of an arbitrary fundamental order.

Let $\mathcal{AP}_T^{\mathcal{D}}$ be the ideal formed by decidable models of an AL theory T.

Theorem 2 (A. Gavryushkin and B. Khoussainov [2]). *Let $\mathcal{L}$ be a finite lattice and $\mathcal{L}'$ be its ideal. Then there exists an AL theory T such that:*

(1) *The fundamental order of T is $\mathcal{L}$, that is, $(\mathcal{AP}_T/\sim, \preceq) \cong \mathcal{L}$.*
(2) *The spectra of decibel models of T is $\mathcal{L}'$, that is, $\mathcal{AP}_T^{\mathcal{D}} \cong \mathcal{L}'$.*

Computable models behave rather differently than decidable ones. I conclude with a theorem that shows that computable models do not form an ideal in the class of all models of an AL theory.

Theorem 3 (A. Gavryushkin and B. Khoussainov [2]). *For every finite lattice $\mathcal{L}$, there exists a theory T of finite signature with countably many models such that:*

(1) *The fundamental order of T without the least element is isomorphic to $\mathcal{L}$.*
(2) *For all $p \in \mathcal{L}$, the class of models corresponding to p contains infinitely many models of which exactly one is computable.*

References

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